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Young tableaux, Schur functions and SU(2) plethysms

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Abstract. Young tableaux and Schur functions are used to analyse the angular momentum eigenstates of multiparticle configurations. The method is based on the plethysms governing the restriction from SU$(2j+1)$ to SU(2). Certain conjectured identities between such SU(2) plethysms are proved and generalised. Fixed symmetry generating functions are specified, and some infinite Schur function series are introduced and a connection is made with Macdonald's identities.

1. Introduction

It is well known that the classification of the various angular momentum eigenstates of a multiparticle configuration of identical electrons or nucleons can be accomplished group theoretically (Racah 1949, Jahn 1950, Flowers 1952, Judd 1962, Wybourne 1970). The essence of this problem is the decomposition into irreducible representations $\{l\}$ of the representation $\{m\}\otimes\{\nu\}$ of SU(2) in accordance with the formula

$$\{m\}\otimes\{\nu\}=\sum_{l} G^{l}_{\nu m}\{l\}. \tag{1.1}$$

The notation adopted here is that of Littlewood (1950a, b), whereby $\{m\}\otimes\{\nu\}$ is the plethysm corresponding to that part of the nth tensor power of the irreducible representation $\{m\}$ of SU(2) whose symmetry is specified by the irreducible representation (ν) of the symmetric group, S_n, acting on the factors constituting the nth power. Thus $\nu=(\nu_1, \nu_2, \ldots)$ signifies a partition of n. The dimension of $\{m\}$ is $m+1$ so that the corresponding value of the angular momentum of the individual particles is $2m$. Similarly $\{l\}$ has dimension $l+1$ and the total angular momentum of the n particle state is $2l$.

The literature (Murnaghan 1954, Hamermesh 1962, Wybourne 1970) contains several tabulations of these plethysms which have been studied in a number of different contexts. For example those plethysms $\{m\}\otimes\{n\}$, for which $\nu=(n)$ is a one part partition, play a central role in invariant theory. A fundamental result in this connection is Hermite's law of reciprocity (Hermite 1854) which states that the number of invariants and covariants of degree m in a binary form of degree n is the same as the number of invariants and covariants of degree n in a binary form of degree m. This corresponds to the identity

$$\{m\}\otimes\{n\}=\{n\}\otimes\{m\}, \tag{1.2}$$

as pointed out by Murnaghan (1951) who went on to give an additional identity

(Murnaghan 1954)

$$\{m\}\otimes\{1^n\}=\{m-n+1\}\otimes\{n\} \tag{1.3}$$

along with the result

$$\{m\}\otimes\{1^n\}=\{m\}\otimes\{1^{m-n+1}\}. \tag{1.4}$$

This latter corresponds in atomic and nuclear spectroscopy to the well known equivalence between particles and holes (Wybourne 1969).

More recently whilst studying generating functions for SU(2) plethysms Patera and Sharp (1981) conjectured the validity of the identities

$$\begin{aligned}&\{y+z-1\}\otimes\{x^y\}=\{y+x-1\}\otimes\{z^y\}=\{z+y-1\}\otimes\{x^z\}\\&=\{x+z-1\}\otimes\{y^x\}=\{z+x-1\}\otimes\{y^z\}=\{x+y-1\}\otimes\{z^x\}.\end{aligned} \tag{1.5}$$

These generalise (1.2)-(1.4), each of which may be recovered from (1.5) with an appropriate choice of x, y and z.

In this paper the validity of (1.5) will be established and generalised still further. This is done not with the intention of producing results of any great physical significance, but with a view to demonstrating the power of Schur function methods. In carrying out this analysis it is advantageous to use a more modern notation. To this end an attempt is made to adhere to the notation of Macdonald (1979) whose text incorporates implicitly various results required by theoretical physicists.

In § 2 the notation for partitions, Young tableaux, plane partitions and Schur functions is introduced, culminating in specific formulae for the plethysms which serve to evaluate (1.1). These formulae are used in § 3 to prove the identities (1.2)-(1.5) which are generalised in § 4. One-to-one correspondences underlying (1.2) and (1.5) are described in § 5 and followed in § 6 by a short discussion and exemplification of fixed symmetry generating functions. A final connection is made in § 7 with certain infinite series of Schur functions and Macdonald's identities (1972).

2. Young tableaux and Schur functions

A partition λ of length $l(\lambda)$ and weight $|\lambda|$ (Macdonald 1979, p 1) is a sequence of $l(\lambda)$ positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{l(\lambda)} > 0$ and $\lambda_1+\lambda_2+\ldots+\lambda_{l(\lambda)}=|\lambda|$. To each such partition there corresponds a Young diagram $F(\lambda)$ consisting of $|\lambda|$ boxes arranged in $l(\lambda)$ left-adjusted rows of lengths $\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}$. Identifying $F(\lambda)$ by the positions of the boxes which define it, we have

$$F(\lambda)=\{(i,j)\,|\,(i,j)\in Z^2, 1\leqslant j<\lambda_i\}. \tag{2.1}$$

The conjugate of a partition λ is the partition λ' whose Young diagram $F(\lambda')$ is obtained from $F(\lambda)$ by interchanging rows and columns. Clearly $l(\lambda')=\lambda_1$, $l(\lambda)=\lambda'_1$, $|\lambda'|=|\lambda|$ and

$$\begin{aligned}F(\lambda')&=\{(i,j)\,|\,(i,j)\in Z^2, (j,i)\in F(\lambda)\}\\&=\{(i,j)\,|\,(i,j)\in Z^2, 1\leqslant j\leqslant \lambda'_i\}.\end{aligned} \tag{2.2}$$

The Frobenius rank $r(\lambda)$ of λ is the number of boxes in the main diagonal of $F(\lambda)$, that is the number of rows of $F(\lambda)$ such that $\lambda_i \geq i$. The Frobenius symbol specifying λ can be written in the form $\lambda = (a|b) = (a_1, a_2, \ldots, a_{r(\lambda)}|b_1, b_2, \ldots, b_{r(\lambda)})$ with $a_k = \lambda_k - k$ and $b_k = \lambda'_k - k$ for $k = 1, 2, \ldots, r(\lambda)$. Thus $a_1 > a_2 > \ldots > a_{r(\lambda)} \geq 0$ and $b_1 > b_2 > \ldots > b_{r(\lambda)} \geq 0$. It follows that $r(\lambda') = r(\lambda)$ and $(\lambda') = (a|b)' = (b|a)$.

It is convenient to introduce the M complement of λ which is the partition λ^* whose Young diagram $F(\lambda^*)$ is obtained by deleting the boxes of $F(\lambda)$ from $F(\lambda_1^M)$, left-adjusting the resulting rows of boxes and then top-adjusting the resulting columns of boxes. Thus

$$\begin{aligned} F(\lambda^*) &= \{(i,j)|(i,j) \in F(\lambda_1^M), (M-i+1, \lambda_1 - j+1) \notin F(\lambda)\} \\ &= \{(i,j)|(i,j) \in Z^2, 1 \leq j \leq \lambda_1 - \lambda_{M-i+1}\} \\ &= \{(i,j)|(i,j) \in Z^2, 1 \leq i \leq M - \lambda'_j\}. \end{aligned} \tag{2.3}$$

To illustrate this definition with an example taken from Macdonald (1971, pp 2, 3), if $\lambda = (54^21)$ and $M = 6$ then $\lambda' = (43^31)$ and $\lambda^* = (5^241^2)$. In Frobenius notation $\lambda = (421|310)$, $\lambda' = (310|421)$ and $\lambda^* = (431|410)$.

The following quantities may be associated with each Young diagram $F(\lambda)$

$$n(\lambda) = \sum_{(i,j)}^{\lambda} (i-1) \tag{2.4}$$

$$\begin{aligned} c(\lambda) = n(\lambda') - n(\lambda) &= \sum_{(i,j)}^{\lambda} c(i,j) = \sum_{(i,j)}^{\lambda} (j-i) \\ &= \sum_{k=1}^{r(\lambda)} \tfrac{1}{2}(a_k - b_k)(a_k + b_k + 1) \end{aligned} \tag{2.5}$$

and

$$H(\lambda) = \prod_{(i,j)}^{\lambda} h(i,j) = \prod_{(i,j)}^{\lambda} (\lambda_i + \lambda'_j - i - j + 1) \tag{2.6}$$

where $c(i,j)$ and $h(i,j)$ are known as the content and hook length, respectively, of the box of $F(\lambda)$ in the position specified by (i,j) (Robinson 1961, p 44, Stanley 1971b, Macdonald 1979, p 9). The notation is such that the sums and products are carried out over all (i,j) specifying the position of a box in $F(\lambda)$.

There exist various arrays consisting of numberings of the boxes of a Young diagram. Amongst these are the Young tableaux $T_a(\lambda)$ obtained by inserting an entry $\eta_a(i,j)$ in the (i,j)th box of $F(\lambda)$ for each (i,j) in $F(\lambda)$, subject to the constraints that each entry is taken from the set $S_M = \{1, 2, \ldots, M\}$ and that the entries are non-decreasing across each row from left to right, and are strictly increasing down each column from top to bottom. Thus

$$T_a(\lambda) = \{\eta_a(i,j)|\eta_a(i,j) \in S_M, (i,j) \in F(\lambda), \eta_a(i,j+1) \geq \eta_a(i,j), \eta_a(i+1,j) > \eta_a(i,j)\} \tag{2.7}$$

where a is an index labelling all possible numberings η_a leading to distinct Young tableaux.

A second numbering scheme leads to plane partitions $P_a(\lambda)$ obtained by inserting an entry $\zeta_a(i,j)$ in the (i,j)th box of $F(\lambda)$ for each (i,j) in $F(\lambda)$, subject to the constraints that each entry is taken from the set $S_N = \{1, 2, \ldots, N\}$ and that the entries are non-increasing across rows from left to right and down columns from top to bottom (Stanley 1971a). Thus

$$P_a(\lambda)=\{\zeta_a(i,j)\,|\,\zeta_a(i,j)\in S_N,\ (i,j)\in F(\lambda),\ \zeta_a(i,j+1)\leqslant\zeta_a(i,j),\ \zeta_a(i+1,j)\leqslant\zeta_a(i,j)\}\tag{2.8}$$

where now the index a is an index labelling all possible numberings ζ_a leading to distinct plane partitions.

The importance of Young tableaux is that they serve to define Schur functions (S functions) (Stanley 1971a)

$$s_\lambda(\boldsymbol{x})=\sum_{T_a(\lambda)}\prod_{(i,j)}^{\lambda}x_{\eta_a(i,j)}=\sum_{T_a(\lambda)}\boldsymbol{x}^{\boldsymbol{m}(a)}\tag{2.9}$$

where

$$\boldsymbol{x}^{\boldsymbol{m}(a)}=x_1^{m_1(a)}x_2^{m_2(a)}\ldots x_M^{m_M(a)}\tag{2.10}$$

with $m_k(a)$ equal to the number of entries $\eta_a(i,j)$ of $T_a(\lambda)$ taking the value k.

The substitutional operation or composition known as plethysm (Littlewood 1950a, p 258), signified by $\circ$ (Macdonald 1979, p 65), is such that

$$s_\nu\circ s_\lambda(\boldsymbol{x})=s_\nu(\boldsymbol{y})\tag{2.11}$$

where the components y_a of $\boldsymbol{y}$ are the summands of $s_\lambda(\boldsymbol{x})$, that is

$$\boldsymbol{y}=(\boldsymbol{x}^{\boldsymbol{m}(1)},\boldsymbol{x}^{\boldsymbol{m}(2)},\ldots,\boldsymbol{x}^{\boldsymbol{m}(N)})\tag{2.12}$$

where N is the number of distinct Young tableaux $T_a(\lambda)$.

In what follows a special case of these expressions is required corresponding to $M=2$ and $\lambda=(m)$. In such a case

$$s_m(x_1,x_2)=x_1^m+x_1^{m-1}x_2+x_1^{m-2}x_2^2+\ldots+x_2^m,\tag{2.13}$$

$N=m+1$ and

$$s_\nu\circ s_m(x_1,x_2)=s_\nu(x_1^m,x_1^{m-1}x_2,\ldots,x_2^m).\tag{2.14}$$

Introducing

$$p=x_1x_2\qquad\text{and}\qquad q=x_1x_2^{-1},\tag{2.15}$$

and making use of a known result for $s_\nu(q^m,q^{m-1},\ldots,1)$ (Stanley 1971b, Macdonald 1979, p 27) it follows that

$$s_m(x_1,x_2)=p^{m/2}(q^{m/2}+q^{m/2-1}+\ldots+q^{-m/2})\tag{2.16}$$

and

$$s_\nu\circ s_m(x_1,x_2)=p^{(m/2)|\nu|}q^{n(\nu)-(m/2)|\nu|}\prod_{(i,j)}^{\nu}\left(\frac{1-q^{m+1+c(i,j)}}{1-q^{h(i,j)}}\right).\tag{2.17}$$

An alternative form of this last expression was given by Littlewood (1950b, p 276) and can be written as

$$s_\nu\circ s_m(x_1,x_2)=p^{(m/2)|\nu|}q^{n(\nu)-(m/2)|\nu|}\prod_{1\leqslant a<b\leqslant m+1}\left(\frac{1-q^{\nu_a-\nu_b-a+b}}{1-q^{-a+b}}\right).\tag{2.18}$$

3. Proof of SU(2) plethysm identities

Returning to the problem motivating this study, Wybourne (1969) has stressed the fact that the SU(2) plethysm (1.1) corresponds precisely to the branching of the irreducible representation $\{\nu\}$ of $SU(m+1)$ into irreducible representations $\{l\}$ of SU(2), where the embedding of SU(2) in $SU(m+1)$ is governed by the existence of the irreducible representation $\{m\}$ of SU(2) of dimension $m+1$.

The connection with the work of § 2 then comes about through noting that the character in the irreducible representation $\{\lambda\}$ of $U(M)$ of a group element having eigenvalues $e^{i\phi_j}$ for $j=1, 2, \ldots, M$ is given by (Littlewood 1950a, p 183)

$$\{\lambda\}=s_\lambda(\boldsymbol{x}) \qquad \text{with} \qquad x_j=e^{i\phi_j} \qquad \text{for} \qquad 1\leqslant j\leqslant M. \tag{3.1}$$

As is customary $\{\lambda\}$ is used to specify both a representation and its character.

It follows from the identification (3.1) and the definition (2.9) that the dimension of the irreducible representation $\{\lambda\}$ of $U(M)$ is given by N, the number of distinct Young tableaux $T_a(\lambda)$. Moreover the corresponding embedding of $U(M)$ in $U(N)$ is such that the irreducible representation $\{\nu\}$ of $U(N)$ has a character, which when evaluated for elements of the subgroup $U(M)$, is given by the plethysm

$$\{\lambda\}\otimes\{\nu\}=s_\nu\circ s_\lambda(\boldsymbol{x}) \qquad \text{with} \qquad x_j=e^{i\phi_j} \qquad \text{for} \qquad 1\leqslant j\leqslant M. \tag{3.2}$$

In the case of the group $SU(M)$ the character of the representation $\{\lambda\}$ is again given by (3.1) subject to the constraint

$$x_1x_2\ldots x_M=\exp[i(\phi_1+\phi_2+\ldots+\phi_M)]=1. \tag{3.3}$$

This constraint then ensures that

$$y_1y_2\ldots y_N=1 \tag{3.4}$$

and there exists an embedding of $SU(M)$ in $SU(N)$ such that the representation $\{\nu\}$ branches to $\{\lambda\}\otimes\{\nu\}$ with character given by (3.2), subject to (3.3).

In the case of SU(2) and the representation $\{m\}$ (3.1) leads to the well known formula

$$\begin{aligned}\{m\}&=s_m(x_1,x_2)=s_m(e^{i\phi_1},e^{i\phi_2})=s_m(e^{i\phi/2},e^{-i\phi/2})\\&=e^{i(m/2)\phi}+e^{i(m/2-1)\phi}+\ldots+e^{-i(m/2)\phi}\\&=e^{ij\phi}+e^{i(j-1)\phi}+\ldots+e^{-ij\phi},\end{aligned} \tag{3.5}$$

where $j=m/2$ is the conventional angular momentum label and ϕ, the class parameter, is an angle of rotation. In this case $N=m+1$.

Reverting to the variables of (2.15)

$$p=1 \qquad \text{and} \qquad q=e^{i\phi} \tag{3.6}$$

so that (3.1) and (3.2) give in conjunction with (2.16), (2.17) and (2.18):

$$\{m\}=q^{m/2}+q^{m/2-1}+\ldots+q^{-m/2} \tag{3.7}$$

$$\{m\}\otimes\{\nu\}=q^{n(\nu)-(m/2)|\nu|}\prod_{(i,j)}^{\nu}\left(\frac{1-q^{m+1+c(i,j)}}{1-q^{h(i,j)}}\right) \tag{3.8}$$

and

$$\{m\}\otimes\{\nu\}=q^{n(\nu)-(m/2)|\nu|}\prod_{1\leqslant a<b\leqslant m+1}\left(\frac{1-q^{\nu_a-\nu_b-a+b}}{1-q^{-a+b}}\right). \tag{3.9}$$

A special case of (3.8) immediately yields

$$\{m\}\otimes\{n\}=q^{-mn/2}\frac{(1-q^{m+1})(1-q^{m+2})\dots(1-q^{m+n})}{(1-q)(1-q^2)\dots(1-q^n)}, \tag{3.10}$$

from which Hermite's law of reciprocity, (1.2), follows at once (Elliott 1895, p 169). Moreover

$$\begin{aligned}\{m\}\otimes\{1^n\}&=q^{n(n-1)/2-mn/2}\frac{(1-q^{m+1})(1-q^m)\dots(1-q^{m-n+2})}{(1-q^n)(1-q^{n-1})\dots(1-q)}\\&=q^{-(m-n+1)n/2}\frac{(1-q^{m-n+2})(1-q^{m-n+3})\dots(1-q^{m+1})}{(1-q^n)(1-q^{n-1})\dots(1-q)}\\&=q^{(m-n+1)(m-n)/2-m(m-n+1)/2}\frac{(1-q^{m+1})(1-q^m)\dots(1-q^{n+1})}{(1-q^{m-n+1})(1-q^{m-n})\dots(1-q)},\end{aligned} \tag{3.11}$$

thus proving the validity of (1.3) and (1.4).

Finally,

$$\begin{aligned}\{y+z-1\}\otimes\{x^y\}&=q^{xy(y-1)/2-(y+z-1)xy/2}\prod_{(i,j)}^{x^y}\left(\frac{1-q^{y+z-i+j}}{1-q^{x+y-i-j+1}}\right)\\&=q^{-xyz/2}G(x,y,z),\end{aligned} \tag{3.12}$$

where, as pointed out by Stanley (1971b), $G(x, y, z)$ is the generating function derived by MacMahon (1916, p 243) for plane partitions $P_a(\pi)$ with $\leq y$ rows and $\leq x$ columns, with largest part $\leq z$. This generating function is necessarily symmetric under all permutations of x, y and z. This can be seen explicitly from the expression (Macdonald 1979, p 48)

$$G(x,y,z)=\prod_{i=1}^{x}\prod_{j=1}^{y}\prod_{k=1}^{z}\left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right). \tag{3.13}$$

Alternatively use may be made of (3.9) to show that

$$\begin{aligned}&\{y+z-1\}\otimes\{x^y\}\\&\quad=q^{-xyz/2}\prod_{i=1}^{x+y+z-1}(1-q^{x+y+z-i})!\\&\qquad\times\frac{\prod_{a=1}^{x-1}(1-q^{x-a})!\prod_{b=1}^{y-1}(1-q^{y-b})!\prod_{c=1}^{z-1}(1-q^{z-c})!}{\prod_{u=1}^{x+y-1}(1-q^{x+y-u})!\prod_{v=1}^{y+z-1}(1-q^{y+z-v})!\prod_{w=1}^{z+x-1}(1-q^{z+x-w})!},\end{aligned} \tag{3.14}$$

where

$$(1-q^m)!=(1-q^m)(1-q^{m-1})\dots(1-q).$$

The limit as $q\to 1$ of (3.13), which gives the dimension of the representation $\{x^y\}$ in the group $\mathrm{SU}(y+z)$, formed part of the original plausibility argument for the conjectures (1.5) (Patera and Sharp 1981). It is now clear from (3.14), or equivalently from (3.12) with $G(x, y, z)$ given by (3.13), that $\{y+z-1\}\otimes\{x^y\}$ is totally symmetric under permutations of x, y and z. This proves the validity of (1.5).

4. Further SU(2) plethysm identities

It should be noted that all the identities (1.2)-(1.5) can be generated from the formulae

$$\{m\}\otimes\{\nu\}=\{m\}\otimes\{\nu^*\} \tag{4.1}$$

and

$$\{m\}\otimes\{\nu\}=\{n\}\otimes\{\nu'\} \tag{4.2}$$

where ν^* is the $(m+1)$-complement of ν, and ν' is the conjugate of ν. It is of interest to explore the range of validity of these formulae and in the case of (4.2) to determine the appropriate relationship between m, n and ν.

In fact (4.1) is valid for all partitions ν and corresponds to the well known equivalence between particles and holes in atomic and nuclear spectroscopy (de Shalit and Talmi 1963). To establish this result it is only necessary to consider the relationship between the Young tableaux $T_a(\nu)$ and $T_a(\nu^*)$ of ν and ν^*. It is easy to see that there exists a one-to-one map from $T_a(\nu)$ to $T_a(\nu^*)$. This is constructed as follows. If the entries in the jth column of $T_a(\nu)$ constitute the set $S_j=\{\eta_a(i,j)\,|\,1\leqslant i\leqslant\lambda'_j\}$ then the corresponding complementary Young tableau $T_a(\nu^*)$ is constructed by placing in the (ν_1-j+1)th column entries taken from the set S_j^* complementary to S_j in the set $S=\{1,2,\ldots,m+1\}$. Carrying out this operation for all $j=1,2,\ldots,\nu_1$ gives the required map from $T_a(\nu)$ to $T_a(\nu^*)$. In the case $\nu=(32)$ and $m=3$ this is exemplified by the correspondence

$$T(\nu)=\begin{array}{l}123\\24\end{array}\leftrightarrow T(\nu^*)=\begin{array}{l}113.\\234\\4\end{array}$$

More generally this argument leads to the result

$$s_\lambda(\boldsymbol{x})=(x_1x_2\ldots x_M)^{\lambda_1}s_{\lambda^*}(\boldsymbol{x}^{-1}) \tag{4.3}$$

where $\boldsymbol{x}=(x_1x_2,\ldots,x_M)$ and $\boldsymbol{x}^{-1}=(x_1^{-1},x_2^{-1},\ldots,x_M^{-1})$. As a special case

$$s_m(x_1,x_2)=(x_1x_2)^m s_m(x_1^{-1},x_2^{-1}). \tag{4.4}$$

Therefore from (2.11)-(2.14)

$$\begin{aligned}s_\nu\circ s_m(x_1,x_2)&=(x_1x_2)^{m|\nu|}s_\nu\circ s_m(x_1^{-1},x_2^{-1})\\&=(x_1x_2)^{m|\nu|}s_\nu(y_1^{-1},y_2^{-1},\ldots,y_{m+1}^{-1})\\&=(x_1x_2)^{m|\nu|}(y_1,y_2,\ldots,y_{m+1})^{-\nu_1}s_{\nu^*}(y_1,y_2,\ldots,y_{m+1})\\&=(x_1x_2)^{m|\nu|-m(m+1)\nu_1/2}s_{\nu^*}\circ s_m(x_1,x_2).\end{aligned} \tag{4.5}$$

Since $(m+1)\nu_1=|\nu|+|\nu|^*$ this result can be expressed in the rather more symmetrical form

$$(x_1x_2)^{-m|\nu|/2}s_\nu\circ s_m(x_1,x_2)=(x_1x_2)^{-m|\nu^*|/2}s_{\nu^*}\circ s_m(x_1,x_2). \tag{4.6}$$

In the case of SU(2) plethysms for which $x_1x_2=1$ this implies the validity of (4.1) for all ν and m with ν^* the $(m+1)$ complement of ν.

As far as (4.2) is concerned the key result (2.17) can be written in the form

$$s_\nu \circ s_m(x_1, x_2) = p^{m|\nu|/2} q^{n(\nu)-m|\nu|/2} \prod_{(i,j)}^{\nu} \frac{1}{(1-q^{h(i,j)})}$$
$$\times \prod_{k=1}^{r(\nu)} (1-q^{m+a_k+1})(1-q^{m+a_k}) \ldots (1-q^{m-b_k+1}) \tag{4.7}$$

where in Frobenius notation $\nu = (a|b)$. In order to determine a relationship of the type (4.2) it is necessary to relate this expression to a similar one appropriate to $\nu' = (b|a)$. However

$$(1-q^{m+a_k+1})(1-q^{m+a_k}) \ldots (1-q^{m-b_k+1})$$
$$= (1-q^{n+b_k+1})(1-q^{n+b_k}) \ldots (1-q^{n-a_k+1}) \quad \text{for } k = 1, 2, \ldots, r(\nu) \tag{4.8}$$

if and only if

$$(n-m) = d = d_k = a_k - b_k \qquad \text{for } k = 1, 2, \ldots, r(\nu). \tag{4.9}$$

Denoting partitions which satisfy this condition by κ rather than ν, so that

$$\kappa = (a_1, a_2, \ldots, a_{r(\kappa)} | a_1 - d, a_2 - d, \ldots, a_{r(\kappa)} - d), \tag{4.10}$$

it is not difficult to see that

$$\tfrac{1}{2}(m-n)|\kappa| - n(\kappa) + n(\kappa') = -\tfrac{1}{2}d|\kappa| + c(\kappa) = 0. \tag{4.11}$$

Thus (4.7) yields the result

$$(x_1x_2)^{-m|\kappa|/2} s_\kappa \circ s_m(x_1x_2) = (x_1x_2)^{-n|\kappa|/2} s_{\kappa'} \circ s_m(x_1x_2). \tag{4.12}$$

Correspondingly

$$\{y+z-1\} \otimes \{\kappa\} = \{x+z-1\} \otimes \{\kappa'\} \tag{4.13}$$

for all z, provided that κ is of the form (4.10) and $x = a_1 + 1$ and $y = b_1 + 1 = a_1 - d + 1$. This is the required generalisation of an identity contained in (1.5) and gives the range of validity of (4.3), namely that specified by (4.9).

5. One-to-one correspondence

The derivation of the result (4.1) presented in § 4 has the merit of involving a one-to-one correspondence between the contributions to $\{m\} \otimes \{\nu\}$ and $\{m\} \otimes \{\nu^*\}$. This explains the existence of the identity by relating it directly to the equivalence between particles and holes, an explanation already given by Wybourne (1969) for the identity (1.4). It would be desirable to establish other one-to-one correspondences explaining the remaining identities of this paper, whether or not they have a physical interpretation.

In the case of (1.2) there is a well known one-to-one correspondence due to Ferrers. The contributions to $\{m\} \otimes \{n\} = s_n \circ s_m(x_1, x_2)$ are found by enumerating the Young tableaux

$$T(n) = \eta(1,1)\eta(1,2) \ldots \eta(1,n) \tag{5.1}$$

with $1 \leqslant \eta(1,1) \leqslant \eta(1,2) \leqslant \ldots \leqslant \eta(1,n) \leqslant m+1$. Setting

$$\lambda_j = m + 1 - \eta(1,j) \tag{5.2}$$

yields a partition λ whose conjugate λ' has parts

$$\lambda'_j = n + 1 - \xi(1, j) \tag{5.3}$$

where $1 \leqslant \xi(1,1) \leqslant \xi(1,2) \leqslant \ldots \leqslant \xi(1,m) \leqslant n+1$. Hence

$$T(m) = \xi(1,1)\xi(1,2)\ldots\xi(1,m) \tag{5.4}$$

is a Young tableaux giving a contribution to $\{n\}\otimes\{m\} = s_m \circ s_n(x_1, x_2)$. These transformations are illustrated in the case $m=6$ and $n=3$ by the example

$$T(n) = 223 \leftrightarrow \lambda = (5^2 4) \leftrightarrow \lambda' = (3^4 2) \leftrightarrow T(m) = 11112. \tag{5.5}$$

The existence of this map, which is one-to-one, provided an early proof of Hermite's law of reciprocity (1.2) (Elliott 1895, p 160).

The generalisation of this result which leads directly to (1.5) comes about by mapping the set of Young tableaux $T_a(x^y)$ contributing to $\{y+z-1\}\otimes\{x^y\} = s_{x^y} \circ s_{y+z-1}(x_1, x_2)$ to the set of plane partitions $P_a(\pi)$ for all partitions π for which $l(\pi) \leqslant y$, $l(\pi') \leqslant x$. The map is such that if $\eta_a(i,j)$ is the (i,j)th entry in $T_a(x^y)$ then the (i,j)th entry of $P_a(\pi)$ is given by

$$\zeta_a(i,j) = z + i - \eta_a(i,j). \tag{5.6}$$

Notice that (5.2) is just a special case of this with $i=1$. Taking $x=3$, $y=2$ and $z=4$ the map from $T_a(x^y)$ to $P_a(\pi)$ is illustrated by

$$T_a(3^2) = \begin{matrix}112\\246\end{matrix} \leftrightarrow P_a(32) = \begin{matrix}443\\42\end{matrix} \tag{5.7}$$

where as is usual zero entries of $P_a(\pi)$ are ignored. For given x, y and z such maps are one-to-one.

Associated with any plane partition $P_a(\pi)$ is a three-dimensional partition each of whose entries is 1. This three-dimensional partition has six aspects (MacMahon 1916, p 179) which define six plane partitions. They are obtainable from $P_a(\pi)$ through the action of a group of operations generated by interchanging rows and columns of a plane partition and forming the conjugate of each and every ordinary partition constituting a row of the plane partition (Stanley 1971b). For example the six plane partitions corresponding to (4.7), all related by these operations, are

443	3332	221	44	32	2222
42	2211	221	42	32	2211
		211	3	31	111
		21		21.	

Recalling that in this example $x=3$, $y=2$ and $z=4$ the corresponding contributions to the plethysms of (1.5) are specified by the Young tableaux

112	1112	112	11	12	1111
246	3344	223	24	23	2233
		344	47	35	4445
		456		56.	

The appropriate values of M for these terms are

6	5	6	7	7	5

with each entry η giving rise to a factor

$$x_1^{M-\eta}x_2^{\eta-1}=q^{(M+1)/2-\eta}.$$

The total contribution corresponding to each of these Young tableaux is then seen to be q^5, confirming not only that the terms are in a one-to-one correspondence but also that they give identical contributions to the plethysms (1.5).

Unfortunately it does not seem possible to give a physical interpretation of this one-to-one correspondence nor does it seem easy to establish any one-to-one correspondence underlying the more general result (4.13).

6. Generating functions

Whilst the emphasis has been placed here on various relationships between plethysms it should be emphasised that all SU(2) plethysms $\{m\}\otimes\{\nu\}$ may be evaluated using (3.8) or the equivalent formula obtainable from (2.18) (Prasad *et al* 1974). In fact (3.7) with m replaced by l gives

$$\{l\}=q^{-l/2}\left(\frac{1-q^{l+1}}{1-q}\right), \tag{6.1}$$

so that the plethysm coefficients appearing in (1.1) are given by

$$G_{\nu m}^{l}=\left|q^{n(\nu)-(m/2)|\nu|+l/2}(1-q)\prod_{(i,j)}^{\nu}\left(\frac{1-q^{m+1-i+j}}{1-q^{h(i,j)}}\right)\right|_0 \tag{6.2}$$

where $|\ldots|_0$ signifies the coefficient of q^0 in the expression.... It then follows that the fixed symmetry generating functions introduced by Patera and Sharp (1981) are given by

$$\begin{aligned} G_\nu(M,L)&=\sum_{l,m}G_{\nu m}^{l}M^mL^l\\ &=\left|\frac{(1-q)}{(1-q^{1/2}L)}\prod_{(i,j)}^{\nu}\frac{1}{(1-q^{h(i,j)})}\right.\\ &\quad\left.\times\left(\sum_{m=0}^{\infty}M^mq^{n(\nu)-(m/2)|\nu|}\prod_{(i,j)}^{\nu}(1-q^{m+1-i+j})\right)\right|_0. \end{aligned} \tag{6.3}$$

By way of illustration, if $\nu=(2^2)$ the factors $m+1-i+j$ and $h(i,j)$ are $m(m+1)$ and 32, $(m-1)m$ and 21 respectively, whilst $|\nu|=4$ and $n(\nu)=2$.

Thus

$$\begin{aligned} G_{2^2}(M,L)&=\left|\frac{1}{(1-q^{1/2}L)}\frac{1}{(1-q^2)^2(1-q^3)}\right.\\ &\quad\left.\times\sum_{m=0}^{\infty}[M^mq^{2-2m}(1-q^{m+1})(1-q^m)^2(1-q^{m-1})]\right|_0 \end{aligned}$$

$$= \left| \frac{1}{(1-qL^2)} \frac{q^2}{(1-q^2)^2(1-q^3)} \left(\frac{1}{(1-q^{-2}M)} - \frac{(1+q^2)^2}{(1-q^{-1}M)} \right) \right|_0$$

$$= M(1+M^3L^6)/(1-ML^4)(1-M^2L^4)(1-M^3)(1-M), \tag{6.4}$$

in agreement with the result of Patera and Sharp (1981).

7. Schur function series

The set of partitions $\kappa = (a|b)$ which satisfy (4.9) for some fixed integer d together with the partition (0) may be used to specify a family of infinite S function series

$$K_d = \sum_{\kappa} (-1)^{k(\kappa)} s_\kappa(x) \tag{7.1}$$

which for various d and $k(\kappa)$ have already appeared in the literature. To be precise

for $k(\kappa) = \lvert\kappa\rvert/2$	$K_{-1} = A,$
for $k(\kappa) = \lvert\kappa\rvert/2$	$K_{+1} = C,$
for $k(\kappa) = \frac{1}{2}(\lvert\kappa\rvert + r(\kappa))$	$K_0 = E,$
for $k(\kappa) = \frac{1}{2}(\lvert\kappa\rvert - r(\kappa))$	$K_0 = G,$

where A, C, E and G are infinite S function series which appear in a study of branching rules for various group-subgroup décompositions (King 1975) and in a corresponding analysis of root systems (Macdonald 1981, p 46).

The constraints (4.9) appropriate to the partitions (κ) are such that a number of quantities of interest associated with group representations specified by κ may be readily evaluated. Perhaps most striking of all the second Casimir operator eigenvalues are given for each of the classical groups by

$$\mathrm{U}(N) \qquad C_2\{\kappa\} = (1/2N)(N+d)|\kappa| \tag{7.2}$$

$$\mathrm{SU}(N) \qquad C_2\{\kappa\} = (1/2N)[(N+d)|\kappa| - (1/N)|\kappa|^2] \tag{7.3}$$

$$\mathrm{O}(N) \qquad C_2[\kappa] = [1/2(N-2)](N+d-1)|\kappa| \tag{7.4}$$

$$\mathrm{Sp}(N) \qquad C_2\langle\kappa\rangle = [1/2(N+2)](N+d+1)|\kappa|. \tag{7.5}$$

Furthermore the dimensions of the corresponding irreducible representations are given by (El Samra and King 1979)

$$\mathrm{U}(N) \text{ and } \mathrm{SU}(N) \qquad D_N(\kappa) = \prod_{(i,j)}^{\kappa} (N-i+j)/H(\kappa) \tag{7.6}$$

$$\mathrm{O}(N) \qquad D_n[\kappa] = \prod_{\substack{(i,j)\\ i\geqslant j}}^{\kappa} (N+d-1+h(i,j)) \prod_{\substack{(i,j)\\ i<j}}^{\kappa} (N+d-1-h(i,j))/H(\kappa) \tag{7.7}$$

$$\mathrm{Sp}(N) \qquad D_n\langle\kappa\rangle = \prod_{\substack{(i,j)\\ i>j}}^{\kappa} (N+d+1+h(i,j)) \prod_{\substack{(i,j)\\ i\leqslant j}}^{\kappa} (N+d+1-h(i,j))/H(\kappa) \tag{7.8}$$

where $H(\kappa)$ is defined by (2.6).

As a small application of these points it is worth noting that the infinite series K_d appear implicitly in Macdonald's identities (Macdonald 1972). For example in the case of the root system B_l associated with the group SO$(2l+1)$

$$\eta(X)^{2l^2+l} = \prod_{n=1}^{\infty} (1-X^n)^{2l^2+l} = \sum_{\alpha\in A} (-1)^{|\alpha|/2} D_{2l+1}[\alpha] X^{C_2[\alpha]}. \tag{7.9}$$

Since in this case we have $A = K_{-1}$ and therefore $d=-1$ it follows that:

$$\eta(X)^{2l^2+l} = \sum_{\alpha\in A} (-1)^{|\alpha|/2} D_{2l+1}[\alpha] X^{|\alpha|/2}. \tag{7.10}$$

In evaluating this expression all terms α appearing in the infinite series are to be included whether or not they specify irreducible representations of SO$(2l+1)$. However, this raises no problem since the corresponding dimensions $D_{2l+1}[\alpha]$ are defined in all cases by (7.7) with $d=-1$ and $N=2l+1$. It is not necessary to use modification rules to convert $[\alpha]$ to a standard representation label (King 1975) although they may be used to recover from (7.9) the sums of those particular representation labels $v=(v_1 v_2 \ldots v_l)$ specified by Macdonald (1972).

Similar results apply to several other of Macdonald's identities.

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